

Fourth-order flows in surface modelling

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Abstract

This short article is a brief account of the usage of fourth-order curvature flow in surface modelling.

1 Introduction

Recent developments on so-called G^1 surface modelling have been investigated [33] with the aid of fourth order geometric flows. In particular, the surface diffusion flow and the Willmore flow have found extensive application. These equations are geometric, in the sense that they do not depend on the choice of coordinates. Classical problems to which these can be applied include surface blending, N -sized hole filling, and free-form surface fitting. This last application is where the G^1 boundary conditions arise.

Earlier work has used the mean curvature flow to attack problems related to those above. The mean curvature flow is a quasilinear second order geometric evolution equation, which means again that the equation is invariant under choice of coordinates. The mean curvature flow is the steepest descent L^2 -gradient flow of the area functional, and has been investigated analytically at least since Mullins [12]. A landmark result is that of Huisken [5]. Applications of the mean curvature flow to smooth or fair noisy images is very efficient – see [3, 15] for example.

Fourth order curvature flows and geometric differential operators have also been investigated for some time, at least since Mullins [13], who proposed the surface diffusion flow. Analytic results on the surface diffusion flow came quickly. We just mention here [18, 21, 11] and the references contained therein. The Willmore flow was only proposed relatively recently [7] but has received a lot of attention in computer graphics. Other fourth order geometric operators such as the biharmonic operator have also been investigated [1, 2]. We have included further references to the relevant literature in the bibliography.

2 The Models

We consider three models: the surface diffusion flow, the Willmore flow, and the quasi-surface diffusion or ‘naïve biharmonic’ flow [24, 30]. This last one is espe-

cially interesting as it has promise in surface modelling [33] but the theoretical analysis is still wide open.

We do not survey the literature any more here and refer to the introduction for relevant papers, along with the references contained within those.

2.1 Surface diffusion flow

The surface diffusion flow is the natural fourth-order version of mean curvature flow. It is a family $\{M(t)\}_{t>0}$ of closed surfaces evolving according to

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -(\Delta H)n \\ M(0) &= M_0 \\ \partial M(t) &= \Gamma\end{aligned}$$

where Δ is the Laplace-Beltrami operator on $M(t)$ and H , n are the mean curvature and the surface normal of $M(t)$ respectively. We note that

$$\frac{d}{dt} \text{Area} (M(t)) = - \int_{M(t)} |\nabla H|^2 dS$$

where ∇ is the connection on $M(t)$ and dS is the surface measure. Also

$$\frac{d}{dt} \text{Vol} (M(t)) = 0.$$

These properties are interesting for surface modelling.

2.2 Willmore flow

The Willmore flow is the gradient flow of the L^2 -norm of the mean curvature squared. It is a family of surfaces evolving according to

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -(\Delta H + 2H(H^2 - K))n \\ M(0) &= M_0 \\ \partial M(t) &= \Gamma\end{aligned}$$

where $K = \det A$ and A is the second fundamental form, is the Gauss curvature.

For surface modelling, it is important that the Willmore flow reduces the average of the curvature across the whole surface very quickly. This makes it almost ideal for smoothing purposes, where we consider the flow for a very short time only.

2.3 Quasi surface diffusion or biharmonic flow

The (QSDF) is given by

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -(\Delta^2 \rho) \\ M(0) &= M_0 \\ \partial M(t) &= \Gamma\end{aligned}$$

where Δ is the Laplace-Beltrami operator on $M(t)$ and H , n are the mean curvature and the surface normal of $M(t)$ respectively. We note that this flow is not volume preserving and is also not reducing surface area quickly like the standard surface diffusion flow.

However the interesting property of (QSDF) is that in [33] it is found to be very effective in G^1 surface modelling. Unfortunately theoretical analysis is not as abundant for this flow as it is for the Willmore and surface diffusion flows.

3 Application of the flows

Let us briefly discuss the application of the (SDF), (WF), and (QSDF) to the problem of smoothing noisy surfaces.

We first note that the short time behaviour of the flows is very similar, so for the smoothing of small regions of rough surfaces they appear to be of equal value. It is the long term evolution which differentiates the three flows rather strikingly. The (WF) reduces curvature quickly and tends to avoid breaking apart the evolving surface, while (SDF) appears to be happy to tear off various pieces.

The (QSDF) is similar to (SDF), except that it does not respect the initial shape so closely. This is because while (SDF) preserves volume, the (QSDF) does not. On the other hand, the (QSDF) is sometimes better behaved, as it reduces the harmonic energy field (also known as the tension field) of ρ very quickly.

3.1 G^1 character of the flows

Each of the (SDF), (QSDF) and (WF) are fourth-order. This means that for the G^1 surface modelling they are ideal and can ensure continuity of the tangent vectors at the polygon boundary.

This is obviously much better for applications than the mean curvature flow or other second order geometric pdes, as they can only ensure G^0 surface modelling.

We plan on investigating even higher order geometric PDEs in the future. These are particularly relevant for modelling of plates in automobiles. The reflective character of the body is determined by the G^2 nature of the plates. If this is not respected then reflections can flip at polygonal boundaries. This requires at least geometric PDE of sixth order.

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